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EXPANSION PROBLEMS WITH IRREGULAR BOUNDARY CONDITIONS.

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THE expansion problems which form the subject of the following discussion are associated with ordinary linear homogeneous differential equations having no singular points in the interval of variation of the independent variable. In nearly all the problems of this sort that have been studied at length, the boundary conditions are of the type which Birkhoff¹ designates as *regular*. A notable exception occurs in the papers in which Liouville² shows that a considerable part of the classical Sturm-Liouville theory of equations of the second order, including the formal expansion of a given function in series of characteristic functions, is capable of extension to certain special systems of higher order. It is readily seen that³ the boundary conditions there considered are irregular.⁴ Consequently the question of the convergence of Liouville's special series, which Liouville himself does not discuss at all, is still left open by Birkhoff's general theory.

The present paper is devoted to a study of this question, to which the author's attention was called by Professor Bôcher. No attempt is made to attain the greatest possible generality in the coefficients of the differential equation; in the matter of boundary conditions, on the other hand, the hypotheses are sufficiently general to include all

¹ Reference will be made to three papers of Birkhoff by means of Roman numerals as follows:

I. Transactions of the American Mathematical Society, **9**, 219-231 (1908).

II. Transactions of the American Mathematical Society, **9**, 373-395 (1908).

III. Rendiconti del Circolo Matematico di Palermo, **36**, 115-126 (1913).

The definition of regular boundary conditions is introduced in II, 382-383.

In connection with these papers of Birkhoff, see also the following articles by Tamarkine:

I. Rendiconti del Circolo Matematico di Palermo, **34**, 345-382 (1912).

II. Rendiconti del Circolo Matematico di Palermo, **37**, 376-378 (1914).

² I. Journal de l'École Royale Polytechnique, **15**, cahier 25, 85-117 (1837).

II. Journal de Mathématiques pures et appliquées, **3**, 561-614 (1838).

³ When the differential equation is of order higher than the second.

⁴ The boundary conditions are given by Liouville in non-homogeneous form, but it is immediately apparent that they are equivalent to a set of homogeneous conditions.

of Liouville's irregular cases and a large class of others. The conclusion reached is not at all that suggested by the analogy of the more familiar expansions. It is found that functions of the sort usually developed in uniformly convergent series of characteristic functions give rise here to series which are very rapidly divergent. The explanation is that while the characteristic functions are essentially of the nature of real trigonometric functions in the cases usually treated, the corresponding functions here involve real exponential factors by which the order of magnitude of the terms of the series is entirely changed.

The problem will be treated, first of all, in one of its simplest particular cases,⁵ which shows clearly the essential difference between the expansions now under consideration and the familiar ones (Section I). Then follows a generalization (Section II) which is more than sufficient to include all of Liouville's irregular boundary conditions. A subsequent further generalization (Section III) illuminates more clearly the relation between these irregular cases and certain regular ones.

I. SPECIAL DIFFERENTIAL EQUATION OF THE THIRD ORDER.

Let us consider the differential equation ⁶

$$(1) \quad \frac{d^3 u}{dx^3} + \rho^3 u = 0$$

in the interval $0 \leq x \leq \pi$, with the boundary conditions

$$(2) \quad u(0) = 0, \quad u'(0) = 0, \quad u(\pi) = 0.$$

The parameter ρ is permitted to take on complex as well as real values. If ω denotes the quantity

$$\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{1}{2}i\sqrt{3},$$

⁵ Cf. Liouville, loc. cit. I; also H. Laudien, Dissertation, Breslau, 1914, pp. 67-90. In the latter memoir, with which the present writer became acquainted only after reaching the main conclusions now expressed, there is an extended discussion of this special case, and it is stated that the development of a function satisfying the usual conditions is uniformly convergent. Laudien appears to have been misled by a false analogy in thinking it allowable, in the demonstration of an auxiliary theorem, to dispense with a careful examination of one of two cases, both of which are essential to the proof of the convergence of the expansion of any one arbitrary function.

⁶ The discussion in the present section has numerous points of contact with the work of Liouville, loc. cit. I, and Laudien, loc. cit.

the general solution of the differential equation, when $\rho \neq 0$, has the form ⁷

$$Ae^{-\rho x} + Be^{-\omega\rho x} + Ce^{-\omega^2\rho x},$$

where A , B , and C are constants. A solution which satisfies the first two of the given boundary conditions for an arbitrary value of ρ is obtained by setting $A = 1$, $B = \omega$, $C = \omega^2$, and except for an arbitrary constant factor this is the only solution which does so; in particular, not more than one characteristic function can correspond to any one characteristic value. A necessary and sufficient condition that $\rho \neq 0$ be a characteristic value is that the solution just mentioned take on the value 0 when $x = \pi$, that is, that ρ satisfy the equation

$$(3) \quad e^{-\rho\pi} + \omega e^{-\omega\rho\pi} + \omega^2 e^{-\omega^2\rho\pi} = 0.$$

As the problem is concerned at the outset only with the cube of ρ , there is no loss of generality in supposing that ρ lies in the sector of the complex plane for which

$$(4) \quad -\frac{\pi}{3} \leq \arg \rho \leq \frac{\pi}{3}.$$

For example, the equation (3) remains unchanged if ρ is replaced by $\omega\rho$, the left-hand member being merely multiplied by ω^2 . It will accordingly be assumed henceforth that (4) is fulfilled, unless the contrary is expressly stated.

If the first term of (3) were absent, the equation would reduce to

$$(5) \quad e^{-\omega\rho\pi} = -e^{\frac{2\pi i}{3}} e^{-\omega^2\rho\pi},$$

which can be solved explicitly by taking the logarithms of both sides. It has the infinitely many roots

$$\rho'_n = \frac{1}{3\sqrt{3}} + \frac{2n}{\sqrt{3}},$$

where n is any integer. Now if ρ is in the sector defined by (4), and large numerically, the first term of (3) is very small, while at least one of the others is large, and it is naturally conjectured, and readily proved, that throughout that part of the sector which lies outside a circle of sufficiently large radius about the origin the roots of (3)

⁷ The general solution for $\rho = 0$ is of course $A + Bx + Cx^2$. It is obvious that $\rho = 0$ is not a characteristic value, since the insertion of this expression in the boundary conditions gives at once $A = B = C = 0$.

closely correspond with those of (5). Let us show that this is the case.⁸

Let ϵ be an arbitrarily small positive quantity, and let R be the region obtained by removing from the whole ρ -plane the interiors of all circles of radius ϵ with centres at the points ρ'_n . If the left-hand member of (3), minus the first term, is denoted by $\varphi(\rho)$, then, in the identity

$$\varphi(\rho) = \omega e^{-\omega\rho\pi} (1 + \omega e^{\sqrt{3}\rho\pi i}),$$

the last factor on the right is a function of ρ having the period $2/\sqrt{3}$. It vanishes only at the points ρ'_n , and becomes infinite uniformly or approaches uniformly the limit 1 as the imaginary part of ρ becomes negatively or positively infinite. In the region R , consequently, its absolute value has a positive minimum δ . The absolute value of the parenthesis in the alternative identity

$$\varphi(\rho) = \omega^2 e^{-\omega^2\rho\pi} (1 + \omega^2 e^{-\sqrt{3}\rho\pi i})$$

has the same minimum δ , for the two parentheses take on conjugate imaginary values for conjugate imaginary values of ρ . Let a circle be described with centre at the origin and radius so large that, in the part of the sector (4) which lies outside it, $|e^{-\rho\pi}| < \frac{1}{2}\delta$; and suppose ρ restricted for the moment to this part of the sector. If ρ is exterior to all the little circles, $\varphi(\rho) + e^{-\rho\pi}$ is surely not zero, because of the inequalities

$$(6) \quad |\varphi(\rho)| > \delta, \quad |e^{-\rho\pi}| < \frac{1}{2}\delta,$$

the first of which is a consequence of the definition of δ and the fact that at least one of the quantities $e^{-\omega\rho\pi}$, $e^{-\omega^2\rho\pi}$, is in absolute value greater than 1. On the other hand, if ρ describes the circumference of one of the little circles in the positive sense, the argument of $\varphi(\rho)$ increases by 2π in a single circuit, since φ has just one simple⁹ root inside, and it follows from the inequalities (6) that the argument of $\varphi(\rho) + e^{-\rho\pi}$ must do likewise. Hence (3) has just one root in each of the little circles. That these roots are actually and not merely asymptotically real, appears from the fact that complex roots of (3)

⁸ The discussion of this point is put in a form adapted to the demonstration of Theorem I; cf. Birkhoff II, pp. 384-386; Tamarkine I, pp. 353-358. It yields more than is needed for the proof of Theorem II, and may be replaced by a simpler argument, as far as that theorem is concerned; see below.

⁹ It is immediately verified that $\varphi(\rho)$ and its derivative can not vanish simultaneously. The same is true of $\varphi(\rho) + e^{-\rho\pi}$ for $\rho \neq 0$, so that all the roots of (3) which give characteristic values, even those in the finite region which has been temporarily left out of account, are of the first order.

must occur in conjugate pairs.¹⁰ Returning to the rejected part of the sector (4), we need not inquire minutely what roots may be present there; it is sufficient for our purposes to recognize that, as φ is analytic in the finite region in question, there can not be infinitely many of them. There will be an index g , positive, negative, or zero, such that if the characteristic numbers¹¹ are arranged in order of increasing absolute value, and the first is denoted by ρ_g , the second by ρ_{g+1} , and so on, the number ρ_n will be nearly equal to ρ'_n , for large values of n . To summarize:

The system (1), (2) has infinitely many characteristic values ρ_n , of which not more than a finite number can be imaginary.¹² The characteristic numbers can be written in the form

$$\rho_n = \frac{1}{3\sqrt{3}} + \frac{2n}{\sqrt{3}} + \epsilon_n, \quad n = g, g+1, g+2, \dots,$$

where

$$\lim_{n=\infty} \epsilon_n = 0.$$

To this we may add at once that there is just one characteristic function $u_n(x)$ corresponding to each characteristic number,

$$(7) \quad \begin{aligned} u_n(x) &= e^{-\rho_n x} + \omega e^{-\omega \rho_n x} + \omega^2 e^{-\omega^2 \rho_n x} \\ &= e^{-\rho_n x} + e^{\frac{1}{2}\rho_n x} \left(\sqrt{3} \sin \frac{\sqrt{3}}{2} \rho_n x - \cos \frac{\sqrt{3}}{2} \rho_n x \right), \end{aligned}$$

the latter expression involving only real quantities if ρ_n is real.

Now if the usual expansion theory were capable of extension to the present problem, it would be possible to develop any function, satisfying certain not very restrictive conditions, in a uniformly convergent series of the form

$$(8) \quad \sum_{n=g}^{\infty} a_n u_n(x).$$

But we shall see that the series is very far from possessing the necessary flexibility. In fact, we shall prove the following theorem:

¹⁰ Cf. Bôcher, Boundary problems in one dimension, International Congress of Mathematicians, Cambridge 1912; see the last paragraph of §6, containing a reference to Birkhoff.

¹¹ If some of the characteristic numbers in the finite region just mentioned should lie on the rays that form the boundary of the sector, we should take into account only the values on one of the rays.

¹² Laudien (loc. cit., pp. 73-76) gives a proof, which he ascribes to Teichmann, that the characteristic values are real from the very beginning.

Theorem I. If a series of the form (8) converges uniformly throughout any interval

$$(9) \quad 0 \leq x < x_0,$$

where $0 < x_0 \leq \pi$, its sum must have continuous derivatives of all orders throughout the interval (9).

Let x_1 and x_2 be any two numbers subject to the inequalities $0 < x_2 < x_1 < x_0$. Since the series converges uniformly throughout the interval (9), it converges uniformly for

$$(10) \quad x_1 \leq x < x_0,$$

and there must be a number G such that ¹³

$$(11) \quad |a_n u_n(x)| < G$$

for all values of x in (10) and for all values of n .

Now if n is sufficiently large, ρ_n will be real, and the interval from $\frac{1}{2}\sqrt{3}\rho_n x_1$ to $\frac{1}{2}\sqrt{3}\rho_n x_0$ will be of length greater than 2π , and so will contain at least one value of $\frac{1}{2}\sqrt{3}\rho_n x$ whose sine is 1. That is, there will be one or more values of x in (10) for which the parenthesis in (7) takes on the value $\sqrt{3}$. Let x'_n be such a value. Then $u_n(x'_n)$ is positive, and

$$u_n(x'_n) > \sqrt{3} e^{\frac{1}{2}\rho_n x'_n} > e^{\frac{1}{2}\rho_n x_1}.$$

Since (11) must be satisfied when $x = x'_n$,

$$|a_n| < G e^{-\frac{1}{2}\rho_n x_1}.$$

On the other hand, if k is any integer,

$$\frac{d^k}{dx^k} u_n(x) = (-\rho_n)^k (e^{-\rho_n x} + \omega^{k+1} e^{-\omega \rho_n x} + \omega^{2k+2} e^{-\omega^2 \rho_n x}),$$

and

$$\left| \frac{d^k}{dx^k} u_n(x) \right| \leq 3\rho_n^k e^{\frac{1}{2}\rho_n x},$$

for any value of x in $(0, \pi)$; we are assuming still that ρ_n is real. If $0 \leq x \leq x_2$, then

$$\left| a_n \frac{d^k}{dx^k} u_n(x) \right| < 3 G \rho_n^k e^{\frac{1}{2}\rho_n(x_2 - x_1)},$$

¹³ Of the assumed uniform convergence, only this consequence will be used; and the statement of the theorem might have been made correspondingly more general.

or, in consequence of the inequalities

$$n < \rho_n < 2n$$

which hold for all values of n from a certain point on,

$$\left| a_n \frac{d^k}{dx^k} u_n(x) \right| < 3 \cdot 2^k G n^k e^{\frac{1}{2}n(x_2 - x_1)}.$$

However large k may be, and however small numerically the negative quantity $x_2 - x_1$, the series which has this last expression for its n th term is convergent. It follows that the series obtained by differentiating (8) term by term k times is uniformly convergent¹⁴ for $0 \leq x \leq x_2$; and as x_2 may be any number between 0 and x_0 , the theorem is proved.¹⁵

The question is now inevitable: Is it possible to develop every *analytic* function in a uniformly convergent series of the form (8), with the restriction, perhaps, that the function shall vanish at the ends of the interval? As in other problems of this sort, it is easy to determine what the values of the coefficients must be, if such an expansion exists. The formal series being given, an answer to the question may be expressed as follows:

Theorem II. If N is any positive integer, however large, it is possible to define a function $f(x)$, analytic throughout the interval $0 \leq x \leq \pi$, and vanishing with its first N derivatives both for $x = 0$ and for $x = \pi$, such that its formal expansion in a series of the form (8) diverges at every interior point of the interval.

The system adjoint to (1), (2) consists of the differential equation

$$-\frac{d^3v}{dx^3} + \rho^3v = 0$$

and the boundary conditions

$$v(0) = 0, \quad v(\pi) = 0, \quad v'(\pi) = 0.$$

Its characteristic numbers are the numbers ρ_n , and its characteristic solutions have the form

$$(12) \quad v_n(x) = e^{\rho_n(x-\pi)} + \omega e^{\omega \rho_n(x-\pi)} + \omega^2 e^{\omega^2 \rho_n(x-\pi)}.$$

¹⁴ Of course the terms of the series are continuous and in fact analytic functions of x .

¹⁵ Professor Birkhoff points out that it may be shown by a slight modification of this argument that the series converges uniformly throughout a region of the complex x -plane including the segment $(0, x_2)$ of the axis of reals in its interior, and so represents a function *analytic* for $0 \leq x < x_0$.

If a function $f(x)$ is the sum of a uniformly convergent series (8), the coefficients in the series must be the numbers

$$(13) \quad a_n = \frac{\int_0^\pi f(x) v_n(x) dx}{\int_0^\pi u_n(x) v_n(x) dx}.$$

The number N being given, let q be the first number greater than or equal to N which is congruent to 2 (mod. 3). We shall examine the terms of the formal series for the special function

$$(14) \quad f(x) = \frac{x^{q+1}(\pi-x)^{q+2}}{\pi^{q+2}(q+1)!}.$$

This function, a polynomial in x , is of course analytic from 0 to π , and vanishes with its first q derivatives at both ends of the interval, while the $(q+1)$ th derivative takes on the value 0 at the point π and the value 1 at the point 0.

To determine the order of magnitude of the numerator in (13), when $f(x)$ is the function just defined, let us break up the integral into three, corresponding to the three terms in the right-hand member of (12), and consider for the moment the second of these. The index n may be supposed so large that ρ_n is real and positive. Integrating by parts $q+2$ times, we find that

$$\begin{aligned} \int_0^\pi f(x) e^{\omega \rho_n(x-\pi)} dx &= \left(\frac{-1}{\omega \rho_n} \right)^{q+2} e^{-\omega \rho_n \pi} \\ &\quad + \left(\frac{-1}{\omega \rho_n} \right)^{q+2} \int_0^\pi f^{(q+2)}(x) e^{\omega \rho_n(x-\pi)} dx, \end{aligned}$$

and hence, by one more integration by parts and the application of obvious inequalities, that

$$\begin{aligned} \left| \int_0^\pi f(x) e^{\omega \rho_n(x-\pi)} dx - \left(\frac{-1}{\omega \rho_n} \right)^{q+2} e^{-\omega \rho_n \pi} \right| &< \frac{c}{\rho_n^{q+3}} \left| e^{-\omega \rho_n \pi} \right| \\ &= \frac{c}{\rho_n^{q+3}} e^{\frac{1}{2} \rho_n \pi}, \end{aligned}$$

where c is independent of n . A corresponding inequality holds for the third integral, with the same value of c . As for the first, it obviously remains finite¹⁶ as n becomes infinite, and so it is certainly true that

$$\left| \int_0^\pi f(x) e^{\rho_n(x-\pi)} dx \right| < \frac{c}{\rho_n^{q+3}} e^{\frac{1}{2} \rho_n \pi}$$

¹⁶ As a matter of fact, of course, it approaches zero.

from a certain point on. When n is large, therefore,

$$\left| \int_0^\pi f(x) v_n(x) dx - \left[\omega \left(\frac{-1}{\omega \rho_n} \right)^{q+2} e^{-\omega \rho_n \pi} + \omega^2 \left(\frac{-1}{\omega^2 \rho_n} \right)^{q+2} e^{-\omega^2 \rho_n \pi} \right] \right| < \frac{3c}{\rho_n^{q+3}} e^{\frac{1}{2} \rho_n \pi}.$$

As $q \equiv 2 \pmod{3}$, the expression in brackets reduces to

$$\left(\frac{-1}{\rho_n} \right)^{q+2} \left(e^{-\omega \rho_n \pi} + e^{-\omega^2 \rho_n \pi} \right) = 2 \left(\frac{-1}{\rho_n} \right)^{q+2} e^{\frac{1}{2} \rho_n \pi} \cos \frac{\sqrt{3}}{2} \rho_n \pi.$$

Now

$$\cos \frac{\sqrt{3}}{2} \rho_n \pi = \cos \left(\frac{1}{6} + n + \frac{\sqrt{3}}{2} \epsilon_n \right) \pi = \pm \cos \left(\frac{1}{6} + \frac{\sqrt{3}}{2} \epsilon_n \right) \pi,$$

$$\lim_{n=\infty} \left| \cos \frac{\sqrt{3}}{2} \rho_n \pi \right| = \frac{\sqrt{3}}{2}.$$

It follows that as soon as n is sufficiently large, the absolute value of the bracket is greater than

$$\frac{3}{2 \rho_n^{q+2}} e^{\frac{1}{2} \rho_n \pi},$$

and hence that from a certain point on

$$\left| \int_0^\pi f(x) v_n(x) dx \right| > \frac{1}{\rho_n^{q+2}} e^{\frac{1}{2} \rho_n \pi}.$$

With regard to the magnitude of the denominator in (13), it suffices to observe that, when ρ_n is real and positive,

$$|u_n(x)| \leq 3 e^{\frac{1}{2} \rho_n x}, \quad |v_n(x)| \leq 3 e^{\frac{1}{2} \rho_n (\pi - x)},$$

and so

$$\left| \int_0^\pi u_n(x) v_n(x) dx \right| \leq 9 \pi e^{\frac{1}{2} \rho_n \pi}.$$

This inequality, with the one obtained for the numerator, shows that as n becomes infinite

$$|a_n| > \frac{c_1}{\rho_n^{q+2}},$$

where c_1 is positive and does not depend on n .

From this relation it is apparent that the general coefficient does not approach zero rapidly enough to enable the terms of the series $\sum a_n u_n(x)$

to remain finite as n becomes infinite. Let x_0 be any particular value in the interval $0 < x_0 < \pi$. Modifying the form of (7), we may write

$$u_n(x_0) = e^{-\rho_n x_0} + 2e^{\frac{1}{2}\rho_n x_0} \sin\left(\frac{\sqrt{3}}{2}\rho_n x_0 - \frac{\pi}{6}\right).$$

The sine in the right-hand member may be zero or nearly zero for some values of n , but there are certainly infinitely many values of n for which this is not the case. For

$$\left(\frac{\sqrt{3}}{2}\rho_{n+1}x_0 - \frac{\pi}{6}\right) - \left(\frac{\sqrt{3}}{2}\rho_n x_0 - \frac{\pi}{6}\right) = x_0 + \frac{\sqrt{3}}{2}(\epsilon_{n+1} - \epsilon_n)x_0,$$

which ultimately becomes practically equal to x_0 , so that the two parentheses on the left-hand side can not both be nearly equal to integral multiples¹⁷ of π . It follows that for infinitely many values of n

$$\left|e^{\frac{1}{2}\rho_n x_0} \sin\left(\frac{\sqrt{3}}{2}\rho_n x_0 - \frac{\pi}{6}\right)\right| > c_2 e^{\frac{1}{2}\rho_n x_0},$$

$$|u_n(x_0)| > c_3 e^{\frac{1}{2}\rho_n x_0},$$

and

$$|a_n u_n(x_0)| > \frac{c_4 e^{\frac{1}{2}\rho_n x_0}}{\rho_n^{q+2}}$$

where c_2 , c_3 , and c_4 are all positive and independent of n . As the right-hand member of the last inequality becomes infinite with n , the proof of Theorem II is complete.

It may be remarked that no use has been made in this demonstration of the fact that *all* the characteristic numbers are given by the asymptotic formula that we have employed. The proof would be unimpaired if there were infinitely many others. It is sufficient to know that there exist an infinite number of real characteristic values distributed in accordance with the formula. And this latter fact is

¹⁷ More precisely, if δ is the smaller of the quantities x_0 , $\pi - x_0$, the index n may be made so large that

$$\frac{\sqrt{3}}{2} \left| \epsilon_{n+1} - \epsilon_n \right| x_0 < \frac{\delta}{2},$$

and then the difference between the two parentheses is between $\frac{1}{2}\delta$ and $\pi - \frac{1}{2}\delta$, and one or the other of the parentheses must itself differ from the nearest integral multiple of π by at least $\frac{1}{4}\delta$.

readily established by writing the characteristic equation in the real form

$$e^{-\rho\pi} + 2e^{\frac{1}{2}\rho\pi} \sin\left(\frac{\sqrt{3}}{2}\rho\pi - \frac{\pi}{6}\right) = 0,$$

and observing the changes of sign of the left-hand member.

II. DIFFERENTIAL EQUATION OF ORDER ν WITH $\nu - 1$ SPECIALIZED BOUNDARY CONDITIONS.

The properties of the series studied in the preceding paragraphs are typical of a large class of expansion problems connected with differential equations of order higher than the second. We shall consider now the differential equation ¹⁸

$$(15) \quad \frac{d^\nu u}{dx^\nu} + p_2(x) \frac{d^{\nu-2}u}{dx^{\nu-2}} + \dots + p_\nu(x) u + \rho^\nu u = 0, \quad \nu \geq 3,$$

where the coefficients p are real or complex functions of the real variable x , continuous with their derivatives of all orders, in an interval which we may still take as the interval $0 \leq x \leq \pi$. The ν linearly independent boundary conditions associated with this equation are to be subjected to the essential restriction that $\nu - 1$ of them involve the values of the solution and its derivatives only at the point 0, while the remaining condition may involve both end-points or the point π alone.¹⁹ These conditions can be reduced by linear combination to the form

¹⁸ It is understood that any or all of the derivatives of orders from 0 to $\nu - 2$ may occur in the equation; only the $(\nu - 1)$ th is assumed to be absent. As is well known, a more general differential equation can be reduced to this form by a change of variables; cf. Birkhoff II, p. 373, footnote. The differential equation which Liouville considers in the paper II is specialized in a different way.

¹⁹ Of course it amounts to the same thing, if a number of the conditions involve the derivatives at π in such a way that these can be eliminated from all but one of the conditions by linear combination. A similar remark is applicable to the cases treated later. The conditions must include at least one which involves the point π , otherwise they are equivalent to the set

$$u(0) = u'(0) = \dots = u^{(\nu-1)}(0) = 0,$$

and no characteristic solutions are possible. In the system treated by Liouville, the ν th condition involves *only* the point π , and the coefficients are subject to further restrictions.

$$\begin{aligned}
 (16) \quad W_s(u) &= u^{(k_s)}(0) + \sum_{j=0}^{k_s-1} \alpha_{sj} u^{(j)}(0) = 0, \quad s = 1, 2, \dots, \nu-1, \\
 W_\nu(u) &= u^{(k_\nu)}(\pi) + \sum_{j=0}^{k_\nu-1} \beta_j u^{(j)}(\pi) + \sum_{j=0}^{\nu-1} \alpha_j u^{(j)}(0) = 0,
 \end{aligned}$$

where

$$\nu-1 \geq k_1 > k_2 > \dots > k_{\nu-1} \geq 0, \quad \nu-1 \geq k_\nu \geq 0;$$

and we shall assume them in this form. It would be possible to simplify them still further, if there were any object in doing so. The coefficients α_{sj} , α_j , β_j , are any real or complex constants, and in particular some or all of them may vanish.

The facts which we shall need concerning the form of solutions of the equation (15), apart from the question of any particular boundary conditions, may be taken from the papers of Birkhoff already referred to. Following his notation, we shall use the symbol $[\alpha]$, where α is any constant, to represent a function of $\rho = re^{i\theta}$, or of ρ and a second variable, which approaches the limit α uniformly with respect to θ , or with respect to θ and the other variable, when r becomes infinite; the same symbol will be used to represent as many different functions of this sort as may occur. Furthermore, the roots of the equation $w^\nu + 1 = 0$ are denoted by $w_1, w_2, \dots, w_\nu$. Then it is possible to assign to each value of ρ a fundamental system of solutions of the equation (15), $y_s(x, \rho)$, $s = 1, 2, \dots, \nu$, in such a way that

$$(17) \quad y_s(x, \rho) = e^{\rho w_s x} [1],$$

and at the same time

$$(18) \quad \frac{d^k}{dx^k} y_s(x, \rho) = (\rho w_s)^k e^{\rho w_s x} [1], \quad k = 1, 2, \dots, \nu-1.$$

These solutions have continuous derivatives of all orders with regard to x , for each value of ρ . If l is any one of the numbers $0, 1, \dots, 2\nu-1$, and ρ is restricted to the sector S_l of the complex plane in which

$$\frac{l\pi}{\nu} \leq \arg \rho \leq \frac{(l+1)\pi}{\nu},$$

the choice of this fundamental system may be so made that the functions $y_s(x, \rho)$ and their derivatives with regard to x shall be analytic functions of ρ for each value of x , at least throughout the part of the sector exterior to a circle of sufficiently large radius about the origin.

The last statement is still true if the sector S_i is replaced by another one, bounded by lines parallel to the sides of S_i , and including the latter sector wholly in its interior.²⁰ Cutting off a part of the new sector, if necessary, by a large circle about the origin, we will let T_i stand for the remaining portion, throughout which each function $y_s(x, \rho)$, with its derivatives, is analytic. Of course the regions T_i will overlap each other to some extent, and where they do, two different fundamental systems of solutions will have been defined; but as we shall not have occasion to use the solutions in different regions T_i simultaneously, no confusion will result.

As the problem involves ρ only in the ν th power, attention may be restricted to one ν th of the ρ -plane, consisting, for example, of the sectors S_ν and $S_{2\nu-1}$; if there should be characteristic values on the boundary lines for which

$$\arg \rho = \pm \frac{\pi}{\nu},$$

the values on one of these lines are to be disregarded. We have to deal, then, only with the regions T_0 and $T_{2\nu-1}$, and a third region U which is finite in extent. It follows at once from the general existence theorem that in this finite region a fundamental system of solutions of the differential equation, and so the determinant whose vanishing gives the characteristic values, may be taken as analytic in ρ . The region can contain only a finite number of characteristic values, and these need not concern us further.

Let us consider the region T_0 . Let

$$w_1 = e^{\frac{-\pi i}{\nu}}, \quad w_2 = e^{\frac{\pi i}{\nu}},$$

while $w_3, \dots, w_\nu$, denote the remaining ν th roots of -1 , arranged, for example, in cyclic order, though the convention as to the last $\nu - 2$ subscripts is quite inessential.²¹ Geometrically, the requirement is that when the roots are represented in the complex plane, as the vertices of a regular polygon of ν sides inscribed in the unit circle, with

²⁰ Birkhoff III, p. 117. The assertion is readily justified by replacing the parameter ρ , in the argument of his paper I, by $\rho + \Gamma$, where Γ is a constant.

²¹ Our use of subscripts here is different from Birkhoff's; and in its next stages our discussion, though following his in general method, is vitally affected by the special nature of the boundary conditions.

a vertical side at the extreme right, the lower and upper ends of this side are denoted by w_1 and w_2 respectively. Suppose that θ is a quantity in the interval

$$0 \leq \theta \leq \frac{\pi}{\nu}.$$

The representation of the numbers $w_1 e^{\theta i}, \dots, w_\nu e^{\theta i}$, is obtained by rotating the original polygon through an angle θ . It is clear that as θ varies over the interval, the argument of the complex quantity $(w_1 - w_s) e^{\theta i}$, where s is any subscript greater than 2, has a positive minimum. The latter statement remains true if θ is allowed to range over an interval

$$-\epsilon \leq \theta \leq \frac{\pi}{\nu} + \epsilon,$$

where ϵ is a sufficiently small positive quantity. Now if $\rho = r e^{\theta i}$ is in T_0 , θ does satisfy the last inequalities, at least when r is sufficiently large, and this has the consequence that as r becomes infinite, the real part of $(w_1 - w_s) e^{\theta i}$ remains uniformly away from zero, the real part of $\rho a (w_1 - w_s)$ becomes uniformly infinite, if a is any positive real constant, and $e^{\rho a (w_s - w_1)}$ uniformly approaches zero. Furthermore, the real part of $w_1 e^{\theta i}$ itself remains uniformly away from zero, and $e^{-\rho w_1 a}$ approaches zero uniformly, even when multiplied by any positive power of ρ . These facts will presently find application.

The determinant $\Delta(\rho)$, by the vanishing of which the characteristic numbers are recognized, has for the element in its s -th row and t -th column the quantity $W_s(y_t)$. By substitution of (17) and (18) in (16), it is found that

$$\begin{aligned} W_s(y_t) &= (\rho w_t)^{k_s} [1], \quad s = 1, 2, \dots, \nu - 1, \\ W_\nu(y_t) &= (\rho w_t)^{k_\nu} e^{\rho w_t \pi} [1] + \alpha_\kappa (\rho w_t)^\kappa [1], \end{aligned}$$

where α_κ is the last coefficient a_j that is different from zero in (16); if every a_j is zero, α_κ is understood to have the value zero likewise. The factor ρ^{k_s} , which is common to all the elements of the s -th row, $s = 1, 2, \dots, \nu - 1$, does not affect the vanishing of Δ , and may be divided out; and we will divide out also a factor $\rho^{k_\nu} e^{\rho w_1 \pi}$ from the elements of the last row, though this factor is not explicitly apparent in all the terms. The problem is thereby reduced to that of the vanishing of the determinant

$$\Delta_1 = \begin{vmatrix} w_1^{k_1} [1] & w_2^{k_1} [1] & \dots & w_\nu^{k_1} [1] \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{k_{\nu-1}} [1] & w_2^{k_{\nu-1}} [1] & \dots & w_\nu^{k_{\nu-1}} [1] \\ \left\{ \begin{array}{l} w_1^{k_\nu} [1] \\ + a_\kappa \rho^{\kappa-k_\nu} w_1^\kappa e^{-\rho w_1 \pi} [1] \end{array} \right\} & \left\{ \begin{array}{l} w_2^{k_\nu} e^{\rho \pi (w_2 - w_1)} [1] \\ + a_\kappa \rho^{\kappa-k_\nu} w_2^\kappa e^{-\rho w_1 \pi} [1] \end{array} \right\} & \dots & \left\{ \begin{array}{l} w_\nu^{k_\nu} e^{\rho \pi (w_\nu - w_1)} [1] \\ + a_\kappa \rho^{\kappa-k_\nu} w_\nu^\kappa e^{-\rho w_1 \pi} [1] \end{array} \right\} \end{vmatrix};$$

the exponent $\kappa - k_\nu$ in the last row of this determinant may be positive, negative, or zero.

It is at this point that the import of the remarks made at the close of the last paragraph but one becomes apparent. As ρ becomes infinite in the region T_0 , the first element in the last row of Δ_1 approaches uniformly the limit $w_1^{k_\nu}$, and the elements from the third to the last uniformly approach zero. The second of the two terms which make up the second element of the row is also uniformly infinitesimal.

Let us expand the determinant according to the elements of the last row, and examine the cofactor of the first element. If we suppose each bracket symbol replaced for the moment by unity, the cofactor is a determinant whose value is surely different from zero, not because its rows consist of like powers of a set of distinct quantities (it is not assumed that all the exponents k_s are successive integers), but because, after division of the s -th row by $w_2^{k_s}$, $s = 1, 2, \dots, \nu - 1$, an operation which of course does not affect the vanishing of the determinant, each column may be regarded as consisting of like powers of the $\nu - 1$ distinct quantities $w_2^{2k_1}, w_2^{2k_2}, \dots, w_2^{2k_{\nu-1}}$, the exponents of these powers in the several columns being the integers from 0 to $\nu - 2$ inclusive, in some order; the numbers $w_3/w_2, \dots, w_\nu/w_2$, are even powers of w_2 . If the brackets are restored, the cofactor is a function which uniformly approaches a limit different from zero when ρ becomes infinite. Similar reasoning, with a like conclusion, applies to the cofactor of the second element. Representing the limiting values of the cofactors of the first and second elements by δ_1 and δ_2 respectively, we see that Δ_1 has the form

$$(19) \quad \Delta_1 = [\delta_1 w_1^{k_\nu}] + [\delta_2 w_2^{k_\nu}] e^{\rho \pi (w_2 - w_1)};$$

the terms in the expansion which involve the last $\nu - 2$ elements and the second part of the second element have been merged here in the first bracket symbol.

If Δ_2 is the expression which results when the bracket expressions in (19) are replaced by their limiting values, the equation $\Delta_2 = 0$ can

be solved explicitly, having infinitely many roots given by the relation

$$\rho_n' \pi (w_2 - w_1) = \log \left(-\frac{\delta_1 w_1^{k\nu}}{\delta_2 w_2^{k\nu}} \right) + 2n\pi i,$$

where the first term on the right is understood to refer to a particular determination of the logarithm, and n is any integer. It appears from the determinant expressions for δ_1 and δ_2 that each of these numbers is either the conjugate or the negative of the conjugate of the other; consequently they have the same absolute value, $\delta_1 w_1^{k\nu}$ and $\delta_2 w_2^{k\nu}$ likewise, and the above logarithm is a pure imaginary. Hence, noting that

$$w_2 - w_1 = 2i \sin \frac{\pi}{\nu},$$

we may write

$$\rho_n' = \frac{\mu_0 + n}{\sin (\pi/\nu)},$$

where μ_0 is a real constant. These roots are all of the first order, for the derivative of Δ_2 never vanishes. The passage from the roots of Δ_2 to those of Δ_1 is effected by reasoning of the same sort as that used in connection with the simple case of the third order. The conclusion is as follows:

The equation $\Delta = 0$ has an infinite number of roots, which can be written in the form

$$(20) \quad \rho_n = \frac{\mu_0 + n + \epsilon_n}{\sin (\pi/\nu)}, \quad n = g, g+1, g+2, \dots,$$

where g is some integer, positive, negative, or zero,²² and ²³

$$(21) \quad \lim_{n=\infty} \epsilon_n = 0.$$

With the exception of a finite number at most, the roots are of the first order.

Of course it still remains to consider the possibility of the existence of roots in the region $T_{2\nu-1}$. But an argument analogous to that just completed shows that, at least from a certain point on, this region can

²² Of course it would be possible to adjust μ_0 so as to make g equal to zero.

²³ For the sake of convenience, a definition of ϵ_n is given here which does not exactly correspond with that used in the previous special discussion.

contain roots only in the strip along the axis of reals which $T_{2\nu-1}$ has in common with T_0 , so that a finite number of roots at most are to be added to those already found. By a change in the value of g , if necessary, the expression given above can be made to account for all the characteristic values.

It is evident *a priori* that if the coefficients in the differential equation and in the boundary conditions are real, the conjugate of any characteristic value will also be a characteristic value, and it follows that all the characteristic values with the exception of a finite number will be actually and not merely asymptotically real. We do not restrict ourselves to this case, however.

To each characteristic value from a certain point on corresponds just one characteristic function. For,²⁴ as we have already had occasion to observe, the determinant Δ_1 has first minors which are always different from zero when ρ is sufficiently large, and it follows that the same is true of Δ . An expression for the characteristic function $u_n(x)$ may be obtained by replacing the t -th element in the last row of $\Delta(\rho_n)$ by $y_t(x, \rho_n)$, $t = 1, 2, \dots, \nu$; it is seen that the result of subjecting the expression so obtained to any one of the operations W_s , $s = 1, 2, \dots, \nu - 1$, may be written as a determinant in which two rows are alike, while the result of the operation W_ν is the vanishing determinant $\Delta(\rho_n)$ itself. Inasmuch as a factor independent of x is immaterial, a characteristic solution may be obtained equally well by inserting the functions $y_t(x, \rho_n)$ in place of the elements of the last row of Δ_1 . It is the latter more convenient form of solution that we

²⁴ As a matter of fact, the mere observation that the characteristic value is a root of the first order for Δ suffices to show that there can not be more than one characteristic function; cf., e. g., Goursat, *Annales de la Faculté des Sciences de l'Université de Toulouse*, deuxième série, **10**, 5-98 (1908); pp. 37-38; Birkhoff III, p. 118. A proof is as follows: The derivative of Δ with regard to ρ may be written as a sum of ν determinants, in each of which $\nu - 1$ rows are identical with the corresponding rows of Δ , while the elements of the remaining row are the derivatives of the corresponding elements of Δ . The cofactors of these derivatives in the several determinants are of course first minors of Δ , except as to algebraic sign. If ρ_n is a characteristic value to which two linearly independent characteristic functions correspond, the first minors of Δ are all zero for $\rho = \rho_n$, and the derivative of Δ consequently vanishes. That is, there can not be two independent characteristic functions for ρ_n unless ρ_n is at least a double root of Δ . This general fact will be assumed in the later discussion of other sets of boundary conditions. It is clear that similar reasoning shows that for a root of any order of multiplicity the number of linearly independent characteristic functions can never be greater than the order indicates; cf. Goursat, loc. cit., p. 44. I do not know whether the proof indicated here is to be found in the literature or not; it is the one which Professor Bôcher gives in his lectures.

shall use. Or, rather, we shall use this form if the numbers δ_1 and δ_2 , which were introduced earlier and figure now in the coefficients of y_1 and y_2 , are conjugate imaginaries, but shall multiply it by i if δ_1 is the negative of the conjugate of δ_2 . We have then in either case

$$(22) \quad u_n(x) = \sum_{t=1}^{\nu} [d_t] e^{\rho_t w_t x},$$

where d_1 and d_2 , in particular, are conjugate imaginary quantities different from zero. It is clear that if x is positive and n is large, all the terms after the second will be insignificant in comparison with either of the first two. Let

$$d_1 = e^{h_1 - h_2 i}, \quad d_2 = e^{h_1 + h_2 i},$$

where h_1 and h_2 are real. Then

$$(23) \quad d_1 e^{\rho_n w_1 x} + d_2 e^{\rho_n w_2 x} = 2 e^{h_1 + \rho_n x \cos(\pi/\nu)} \cos\{h_2 + \rho_n x \sin(\pi/\nu)\}.$$

This identity is of course none the less valid for the circumstance that the numbers ρ_n are not necessarily real.

Suppose that a succession of constants ²⁵ $a_n, n = g, g+1, g+2, \dots$, is such that $a_n u_n(x)$ remains uniformly finite throughout an interval included in $(0, \pi)$, or, in symbols,

$$(24) \quad |a_n u_n(x)| < G$$

for $x_1 < x < x_0$, where $0 < x_1 < x_0 \leq \pi$. It is to be shown that throughout the interval $0 \leq x < x_0$ the series

$$(25) \quad \sum_{n=g}^{\infty} a_n u_n(x)$$

must converge and represent a function possessing derivatives of all orders.

Because of the relations (20) and (21), the difference between the last factor in (23) and the expression

$$(26) \quad \cos\{h_2 + (\mu_0 + n)x\}$$

²⁵ This notation is adequate only if all the characteristic values from the very beginning are simple roots of the determinant equation. In order not to raise questions irrelevant to the main issue, it will be assumed, in the contrary case, that the series (25) begins with such a value of the index n that only simple roots are involved. In the extension of Theorem II, where the formal expansion of a definite function is under discussion, it is possible to assign values to the first terms of the expansion immediately; cf. Birkhoff II, p. 380; but we shall not be in any way directly concerned with these terms.

approaches zero, uniformly for $0 \leq x \leq \pi$, when n becomes infinite. When n is sufficiently large, $h_2 + (\mu_0 + n)x$ will vary by more than 2π as x increases from x_1 to x_0 , and there will surely be an intermediate value $x = x_n'$ for which the expression (26) is equal to 1. If this value is substituted for x in (23), for the successive values of n , the last factor remains from a certain point on numerically greater than a positive constant. Independently of this particular choice of x , the quantity

$$e^{\rho_n x} \cos(\pi/\nu) / e^{(\mu_0 + n)x \cot(\pi/\nu)} = e^{\epsilon_n x \cot(\pi/\nu)}$$

remains numerically greater than a positive constant beyond a certain point. Consequently the whole expression (23) remains for $x = x_n'$ numerically greater than a constant positive multiple of $e^{(\mu_0 + n)x_n' \cot(\pi/\nu)}$, and the same is true of $u_n(x_n')$. The inequality is strengthened if x_n' in the exponent is replaced by x_1 ; it follows from (24) that

$$(27) \quad |a_n| < c' e^{-(\mu_0 + n)x_1 \cot(\pi/\nu)},$$

where c' is independent of n .

We have now to consider the order of magnitude of the derivatives of $u_n(x)$. Suppose x restricted to the interval $0 \leq x \leq x_2$, where $x_2 < x_1$. If k is one of the numbers $0, 1, \dots, \nu - 1$, it is at once deduced from (17) and (18) that

$$(28) \quad \left| \frac{d^k}{dx^k} u_n(x) \right| < c'' n^k e^{(\mu_0 + n)x_2 \cot(\pi/n)},$$

where c'' is independent of n . This inequality may be established step by step for larger values of k , by combining the inequalities already found with the given differential equation (15) and the equations obtained from it by successive differentiation. Different values of the constant c'' will be found for different values of k , but this is immaterial.

Because of (27) and (28), the series

$$\sum a_n \frac{d^k}{dx^k} u_n(x)$$

converges uniformly for $0 \leq x \leq x_2$, and consequently represents a continuous function, inasmuch as the individual terms are continuous. The statement of the hypothesis for any particular value of x_1 implies its validity for any larger value of x_1 which still remains less than x_0 , and hence x_2 may be any positive number less than x_0 . With

the reservation that only characteristic functions corresponding to simple roots of the determinant equation are to be taken into account, we are entitled to make the assertion:

The statement of Theorem I is applicable to the series considered in the present section.

There is a corresponding extension of Theorem II. Let the left-hand member of the differential equation (15), without the term $\rho^\nu u$, be denoted by $L(u)$. The differential expression adjoint to this is

$$M(v) = (-1)^\nu \frac{d^\nu v}{dx^\nu} + (-1)^{\nu-2} \frac{d^{\nu-2}}{dx^{\nu-2}} (p_2 v) + \cdots + p_\nu v,$$

and the equation adjoint to (15) is

$$(29) \quad M(v) + \rho^\nu v = 0.$$

The determination of the adjoint boundary conditions²⁶ must be carried through in some detail.

In Lagrange's identity,

$$vL(u) - uM(v) = \frac{d}{dx} P(u, v),$$

the expression $P(u, v)$ is a bilinear form in $u(x), u'(x), \dots, u^{(\nu-1)}(x), v(x), v'(x), \dots, v^{(\nu-1)}(x)$. If this identity is integrated with regard to x from 0 to π , the resulting expression on the right, which may be denoted by $\left\{ P(u, v) \right\}_{x=0}^{x=\pi}$, is a bilinear form in the 2ν pairs of variables

$$\begin{array}{llll} u_1 = u(0), & u_2 = u'(0), & \cdots & u_\nu = u^{(\nu-1)}(0), \\ u_{\nu+1} = u(\pi), & u_{\nu+2} = u'(\pi), & \cdots & u_{2\nu} = u^{(\nu-1)}(\pi), \\ v_1 = v(0), & v_2 = v'(0), & \cdots & v_\nu = v^{(\nu-1)}(0), \\ v_{\nu+1} = v(\pi), & v_{\nu+2} = v'(\pi), & \cdots & v_{2\nu} = v^{(\nu-1)}(\pi). \end{array}$$

Let η denote the matrix of its coefficients, the coefficient of $u_s v_t$ standing in the s -th row and the t -th column. It is clear from the way in which η was obtained that the last ν elements in each of the first ν rows vanish, and likewise the first ν elements in each of the last ν rows.²⁷ Let the linear forms $W_1(u), \dots, W_\nu(u)$, which occur

²⁶ Cf. Birkhoff II, p. 375; Bôcher, Transactions of the American Mathematical Society, **14**, 403-420 (1913); pp. 404-405. Also, Liouville II, p. 604.

²⁷ As a matter of fact, nearly half of the remaining elements also vanish; but this point is not of importance for the present discussion.

in (16), be supplemented by ν linear forms $W_{\nu+1}(u), \dots, W_{2\nu}(u)$, in such a way that all the 2ν forms are linearly independent. If 2ν linear forms $V_1(v), \dots, V_{2\nu}(v)$ are determined in such a way that

$$\sum_{s=1}^{2\nu} W_s(u) V_s(v) = \begin{cases} P(u, v) & x = \pi \\ & x = 0 \end{cases},$$

the equations ²⁸

$$V_s(v) = 0, \quad s = \nu + 1, \nu + 2, \dots, 2\nu,$$

constitute the boundary conditions adjoint to (16). We will denote the matrix of the forms W by α , and that of the forms V by β . Whatever these matrices may be, the matrix of the bilinear form $\Sigma W_s(u) V_s(v)$ is $\alpha'\beta$, where α' is the conjugate of α , and so the determination of the adjoint conditions amounts to the solution of the matrix equation $\alpha'\beta = \eta$, which gives

$$\beta = (\alpha')^{-1}\eta.$$

Of course the reciprocal of α' is the same as the conjugate of α^{-1} . Let us examine the form of the matrix β .

The choice of the ν forms $W_{\nu+1}, \dots, W_{2\nu}$, is arbitrary, provided that the condition of linear independence is fulfilled. We will choose them in a particularly simple way. We will set $W_{\nu+1}$ equal to that one of the variables $u_1, \dots, u_\nu$, which does not appear as leading term in any of the conditions $W_1, \dots, W_{\nu-1}$, in (16), and will let $W_{\nu+2}, \dots, W_{2\nu}$, be equal respectively to the variables $u_{\nu+1}, \dots, u_{2\nu}$, with the omission of the one which forms the leading term of W_ν . It is clear that if the variables u are expressed in terms of the quantities W by solving the linear equations which give the latter in terms of the former, the first ν of the u 's will be completely determined by the first $\nu - 1$ equations and the $(\nu + 1)$ th, and so will not involve W_ν nor $W_{\nu+2}, \dots, W_{2\nu}$. That is, since the matrix of the new equations is α^{-1} , the last $\nu - 1$ columns of α^{-1} (and the ν th column) contain no elements different from zero in the first ν rows. The last $\nu - 1$ rows of the conjugate of α^{-1} contain no elements different from zero in the first ν columns. Consequently, since the first ν columns of η contain non-vanishing elements only in the first ν rows, the last $\nu - 1$ rows of the product $(\alpha')^{-1}\eta$ will have all their elements in the first ν columns equal to zero. This means that $\nu - 1$ of the adjoint boundary conditions $V = 0$

²⁸ The use of subscripts here is different from Birkhoff's; the change is made for the sake of convenience in using the language of matrices.

involve only the point π and not the point 0. The latter point surely will be involved in the remaining condition of the adjoint set. For²⁹ the ν th row of the product matrix, like the last $\nu - 1$ rows, involves only zeros in the first ν columns, and it follows that the remaining ν rows, the $(\nu + 1)$ th in particular, must contain elements different from zero in these columns, since the determinant of the matrix can not vanish. The adjoint conditions may be simplified by linear combination to a form resembling that of the original conditions in (16). No confusion will be caused if we denote the simplified conditions by $V_s(v) = 0$, $s = 1, 2, \dots, \nu$; the conflicting notation hitherto used will not be referred to again. We may write

$$(30) \quad \begin{aligned} V_1(v) &= v^{(k_1')} (0) + \sum_{j=0}^{k_1'-1} \alpha_j' v^{(j)} (0) + \sum_{j=0}^{\nu-1} \beta_j' v^{(j)} (\pi) = 0, \\ V_s(v) &= v^{(k_s')} (\pi) + \sum_{j=0}^{k_s'-1} \beta_{sj}' v^{(j)} (\pi) = 0, \quad s = 2, 3, \dots, \nu, \end{aligned}$$

where

$$\nu - 1 \geq k_1' \geq 0, \quad \nu - 1 \geq k_2' > k_3' > \dots > k_\nu' \geq 0.$$

For each value of ρ , a fundamental system of solutions of the equation (29), $z_s(x, \rho)$, $s = 1, 2, \dots, \nu$, can be assigned in such a way that

$$(31) \quad \frac{d^k}{dx^k} z_s(x, \rho) = (-\rho w_s)^k e^{-\rho w_s x} [1], \quad k = 0, 1, \dots, \nu - 1.$$

It is important for us now, however, to notice that the bracket symbol may be given a much more precise interpretation than was done before. Both the relations (31) and the relations (17), (18), remain true³⁰ if $[a]$, where a is any constant, is understood to mean an expression of the form

$$a + \frac{\psi_1(x)}{\rho} + \frac{\psi_2(x)}{\rho^2} + \dots + \frac{\psi_{m-1}(x)}{\rho^{m-1}} + \frac{E(x, \rho)}{\rho^m},$$

where m is an integer that may be assigned arbitrarily in advance, the functions $\psi_j(x)$, $j = 1, 2, \dots, m - 1$, are continuous with their derivatives of all orders for $0 \leq x \leq \pi$, and $E(x, \rho)$ is a function of x

²⁹ The statement may also be proved by observing that on the contrary assumption the adjoint conditions could be reduced to the form $v(\pi) = v'(\pi) = \dots = v^{(\nu-1)}(\pi) = 0$, and the adjoint system could have no characteristic values.

³⁰ See Birkhoff, I, II.

and $\rho = re^{\theta i}$ which remains finite uniformly in x and θ when r becomes infinite. We shall use the symbol with this meaning from now on; the functions ψ and E will be different in different cases, of course, and are to be replaced on occasion by quantities independent of the variable x . The solutions described in (31), regarded as functions of x , are continuous with their derivatives of all orders. Their analytic character as functions of ρ need not be further specified, for the determination of the characteristic values does not have to be repeated for the system (29), (30). It follows from a general theorem³¹ that the characteristic values for this system are the same as for the system (15), (16), and that the number of linearly independent characteristic solutions corresponding to any one characteristic number ρ_n is the same for both systems. From a certain point on, the system (29), (30), has just one characteristic solution $v_n(x)$ for each number ρ_n , and the corresponding coefficient in the formal expansion of a function $f(x)$ is

$$(32) \quad a_n = \frac{\int_0^\pi f(x)v_n(x)dx}{\int_0^\pi u_n(x)v_n(x)dx}.$$

We shall show how to define analytic functions $f(x)$ for which the general term $a_n u_n(x)$ does not remain finite.

Except for an arbitrary factor independent of x , which will be chosen later, the characteristic solution $v_n(x)$ may be represented by a determinant made up as follows: The t -th element in the first row is $z_t(x, \rho_n)$, which, by (31), has the form $e^{-\rho_n w_t x}$ [1]. The t -th element in the s -th row, $s = 2, 3, \dots, \nu$, is $V_s(z_t)$, or

$$(-\rho_n w_t)^{k'_s} e^{-\rho_n w_t x} [1].$$

On division of the s -th row by $(-\rho_n)^{k'_s}$, $s = 2, 3, \dots, \nu$, and multiplication of the t -th column by $e^{\rho_n w_t x}$, $t = 1, 2, \dots, \nu$, this determinant is reduced to the form

$$(33) \quad \begin{vmatrix} e^{\rho_n w_1 (\pi - x)} [1] & e^{\rho_n w_2 (\pi - x)} [1] & \dots & e^{\rho_n w_\nu (\pi - x)} [1] \\ w_1^{k'_2} [1] & w_2^{k'_2} [1] & \dots & w_\nu^{k'_2} [1] \\ \cdot & \cdot & \cdot & \cdot \\ w_1^{k'_\nu} [1] & w_2^{k'_\nu} [1] & \dots & w_\nu^{k'_\nu} [1] \end{vmatrix}.$$

Let the limiting values of the cofactors of the first two elements of

³¹ See Birkhoff II, pp. 375–376; Bôcher, loc. cit., p. 407.

the first row be denoted by δ'_1 and δ'_2 . As in the corresponding case which was discussed earlier, it is seen that neither of these numbers is zero, and that each is the conjugate or the negative of the conjugate of the other. We shall denote by $v_n(x)$ in the one case the value of the determinant as it stands, in the other, the determinant multiplied by i . Then we have

$$(34) \quad v_n(x) = \sum_{t=1}^{\nu} [d'_t] e^{\rho_n w_t (\pi - x)},$$

where d'_1 and d'_2 are conjugate imaginary quantities different from zero.

Let us examine the order of magnitude of the denominator of the fraction in (32), using the expressions (22) and (34) for the characteristic functions. In consequence of (20), (21), there is a real number γ , independent of n , such that ³²

$$(35) \quad R(\rho_n w_t) < (\mu_0 + n) \cot(\pi/\nu) + \gamma$$

for $t = 1$ and $t = 2$, and *a fortiori* for $t = 3, \dots, \nu$; the constant $\mu_0 \cot(\pi/\nu)$ might be merged with γ . Hence there exist numbers D , D' , and c , all independent of n , such that

$$|u_n(x)| < D e^{n x \cot(\pi/\nu)}, \quad |v_n(x)| < D' e^{n(\pi - x) \cot(\pi/\nu)},$$

and

$$(36) \quad \left| \int_0^\pi u_n(x) v_n(x) dx \right| < c e^{n\pi \cot(\pi/\nu)}.$$

Now let us define a function $f(x)$ by the formula (14), denoting by q for the present an arbitrary positive integer, and seek information about the corresponding value of the integral in the numerator of (32), with the aid of the expression (34) for $v_n(x)$. By integrating by parts, as at the corresponding point in the preceding section, with attention to the special values of $f(x)$ and its derivatives at 0 and π , and supplementing this process by the application of (20), (21), and (35), it is seen that

$$\left| \int_0^\pi f(x) e^{\rho_n w_1 (\pi - x)} dx - \left(\frac{1}{\rho_n w_1} \right)^{q+2} e^{\rho_n w_1 \pi} \right| < \frac{c_1}{n^{q+3}} e^{n\pi \cot(\pi/\nu)},$$

where c_1 , like every quantity denoted by the letter c with a subscript throughout the remainder of this section, is independent of n .

³² By $R(z)$ is meant the real part of z .

It is a difference between the present problem and the special one treated in the preceding section that the coefficient of $e^{\rho_n w_1(\pi-x)}$ in (34) is not a constant, in general, but an expression of the form

$$d'_1 + \frac{\psi_1(x)}{\rho_n} + \frac{\psi_2(x)}{\rho_n^2} + \dots + \frac{\psi_{m-1}(x)}{\rho_n^{m-1}} + \frac{E(x, \rho)}{\rho_n^m}.$$

The additional terms, however, are readily taken into account. The arbitrary number m has not yet been assigned; let it be set equal to $q+3$. Every one of the functions $f(x) \psi_j(x)$, $j = 1, 2, \dots, m-1$, vanishes with its first q derivatives at the points 0 and π . Hence it may be shown by integrating by parts that

$$\left| \int_0^\pi f(x) \psi_j(x) e^{\rho_n w_1(\pi-x)} dx \right| < \frac{c_2}{n^{q+2}} e^{n\pi \cot(\pi/\nu)}.$$

Since each integral of this sort is multiplied by the reciprocal of at least the first power of ρ_n , and, of course,

$$\left| \int_0^\pi f(x) E(x, \rho) e^{\rho_n w_1(\pi-x)} dx \right| < c_3 e^{n\pi \cot(\pi/\nu)},$$

we see that

$$\left| \int_0^\pi f(x) [d'_1] e^{\rho_n w_1(\pi-x)} dx - d'_1 \left(\frac{1}{\rho_n w_1} \right)^{q+2} e^{\rho_n w_1 \pi} \right| < \frac{c_4}{n^{q+3}} e^{n\pi \cot(\pi/\nu)}.$$

A corresponding inequality holds for the integral involving the second term of (34), the constant c_4 being replaced, if necessary, by a new constant c_5 . For $t \geq 3$, the quantity $e^{\rho_n w_t(\pi-x)}$ remains less in absolute value, as n becomes infinite, than $e^{\rho_n w_1 \pi}$ divided by any power of n , and we may write

$$\left| \int_0^\pi f(x) [d'_t] e^{\rho_n w_t(\pi-x)} dx \right| < \frac{c_6}{n^{q+3}} e^{n\pi \cot(\pi/\nu)}, \quad t=2, 3, \dots, \nu.$$

Hence

$$(37) \quad \left| \int_0^\pi f(x) v_n(x) dx - \left\{ d'_1 \left(\frac{1}{\rho_n w_1} \right)^{q+2} e^{\rho_n w_1 \pi} + d'_2 \left(\frac{1}{\rho_n w_2} \right)^{q+2} e^{\rho_n w_2 \pi} \right\} \right| < \frac{c_7}{n^{q+3}} e^{n\pi \cot(\pi/\nu)}.$$

Let $d'_1 = e^{h'_1 - h'_2 i}$, where h'_1 and h'_2 are real, so that $d'_2 = e^{h'_1 + h'_2 i}$. By using (20) and the relation $w_1 = \cos(\pi/\nu) - i \sin(\pi/\nu)$, the first

term in the braces, without the factor ρn^{-q-2} , may be written in the form

$$(38) \quad e^{n\pi \cot(\pi/\nu)} e^{-n\pi i} e^{h'_1 - h'_2 i + \frac{(q+2)\pi i}{\nu} + (\mu_0 + \epsilon_n)\pi \cot(\pi/\nu) - (\mu_0 + \epsilon_n)\pi i}.$$

The factor $e^{-n\pi i}$, of course, is equal to $(-1)^n$. The third factor approaches a limit $H_1 - H_2 i$ as n becomes infinite. The second term in the braces, without the power of ρn , may be written similarly as $e^{n\pi \cot(\pi/\nu)} e^{n\pi i}$ multiplied by a factor which approaches the limit $H_1 + H_2 i$. Consequently the whole expression in braces is equal to $(-1)^n \rho n^{-q-2} e^{n\pi \cot(\pi/\nu)}$ multiplied by a quantity which approaches $2H_1$. It follows that the absolute value of the expression is from some point on greater than

$$\frac{c_8}{n^{q+2}} e^{n\pi \cot(\pi/\nu)},$$

where c_8 is a positive constant, provided that $H_1 \neq 0$.

The condition that H_1 be equal to zero is that the imaginary part of the limiting value approached by the exponent of the third factor in (38), namely, the expression

$$\frac{(q+2)\pi i}{\nu} - h'_2 i - \mu_0 \pi i$$

be an odd multiple of $\frac{1}{2}\pi i$. There is no apparent reason why this condition should not be satisfied, in particular cases. But it can surely not be satisfied for two successive values of q . We may accordingly adopt the following rule, with the assurance that the indicated choice of q is always possible:

Let N be any positive integer. Then q shall be the first integer greater than or equal to N , for which $H_1 \neq 0$.

Assuming that q is so chosen, we infer from (37) that

$$\left| \int_0^\pi f(x) v_n(x) dx \right| > \frac{c_9}{n^{q+2}} e^{n\pi \cot(\pi/\nu)}$$

from some point on, and from this inequality and (36), that

$$\left| a_n \right| > \frac{c_{10}}{n^{q+2}},$$

where c_9 and c_{10} are positive.

The remainder of the proof that the series diverges need not be elaborated. The details are suggested partly by the special discus-

sion in the first section, and partly by the argument just completed. It is found that if x_0 is any fixed value in the interval $0 < x_0 < \pi$, the absolute value of $u_n(x_0)$ is equal to $e^{n x_0 \cot(\pi/\nu)}$ multiplied by a quantity which may be small for particular values of n , but certainly does not approach zero as a limit when n becomes infinite. It is possible to assign a positive constant c_{11} , such that

$$|u_n(x_0)| > c_{11} e^{n x_0 \cot(\pi/\nu)}$$

for infinitely many values of n . It follows that $an u_n(x_0)$ does not remain finite, and

The statement of Theorem II is valid for the series discussed in the present section.

III. DIFFERENTIAL EQUATION OF ORDER ν WITH FEWER THAN $\nu - 1$ SPECIALIZED BOUNDARY CONDITIONS.

If two of the boundary conditions, instead of a single one, involve the point π , the resulting series still show essentially the same characteristics, provided that the order of the differential equation is greater than 4, so that the specialized conditions remain in the majority. We retain the differential equation (15) of the preceding section, assuming now that $\nu \geq 5$, and associate with it the following boundary conditions:

$$(39) \quad \begin{aligned} W_s(u) &= u^{(k_s)}(0) + \sum_{j=0}^{k_s-1} \alpha_{sj} u^{(j)}(0) = 0, & s = 1, 2, \dots, \nu - 2, \\ W_s(u) &= u^{(k_s)}(\pi) + \sum_{j=0}^{k_s-1} \beta_{sj} u^{(j)}(\pi) + \sum_{j=0}^{\nu-1} \alpha_{sj} u^{(j)}(0) = 0, \\ & & s = \nu - 1, \nu, \end{aligned}$$

where

$$\nu - 1 \geq k_1 > k_2 > \dots > k_{\nu-2} \geq 0, \quad \nu - 1 \geq k_{\nu-1} > k_\nu \geq 0.$$

The differential equation being the same, the asymptotic expressions (17), (18), are still available,³³ and the regions S_l and T_l may be defined as before. In determining the distribution of the characteristic values, it is sufficient to consider those in the sectors S_0 and $S_{2\nu-1}$; all but a finite number of these will be in the regions T_0 and $T_{2\nu-1}$. If

³³ For the present we may interpret the bracket symbol as it was interpreted at the beginning of the preceding section, without the refinement which was introduced later.

characteristic values occur on the ray $\arg \rho = \pi/\nu$, which bounds S_0 , the corresponding values on the ray $\arg \rho = -\pi/\nu$ may be left out of account. It will be convenient now to have a definite understanding as to *four* of the subscripts of the roots w_i ; we shall write

$$w_1 = e^{-\frac{\pi i}{\nu}}, \quad w_2 = e^{\frac{\pi i}{\nu}}, \quad w_3 = e^{-\frac{3\pi i}{\nu}}, \quad w_4 = e^{\frac{3\pi i}{\nu}},$$

so that the first four subscripts are assigned to the four right-hand vertices of the ν -sided regular polygon which represents the ν roots. If the polygon is rotated in the positive direction through an angle θ between 0 and π/ν , the vertices representing the numbers $w_1 e^{\theta i}, \dots, w_4 e^{\theta i}$, will be those furthest to the right, in the order of decreasing abscissas. The order of the remaining subscripts is immaterial.

The quantities $W_s(y_i)$ have the asymptotic expressions:

$$\begin{aligned} W_s(y_i) &= (\rho w_i)^{k_s} [1], & s &= 1, 2, \dots, \nu - 2, \\ W_s(y_i) &= (\rho w_i)^{k_s} e^{\rho w_i \pi} [1] + a_s (\rho w_i)^{k_s} [1], & s &= \nu - 1, \nu, \end{aligned}$$

where k_s is the order of the highest derivative whose value at the point 0 actually enters into the s -th condition, for the last two values of s , and a_s is the corresponding coefficient, or is zero if the condition does not involve the point 0 at all. Let the factor ρ^{k_s} be divided from the s -th row of the determinant of these quantities, $s = 1, 2, \dots, \nu$, and in the last two rows let γ_s be written for $k_s - k_{\nu}$. Then the condition for a characteristic value is expressed by the vanishing of a determinant Δ_1 of the following form: ³⁴

$$\begin{vmatrix} w_1^{k_1} [1] & w_2^{k_1} [1] & \dots & w_{\nu}^{k_1} [1] \\ \cdot & \cdot & \cdot & \cdot \\ w_1^{k_{\nu-2}} [1] & w_2^{k_{\nu-2}} [1] & \dots & w_{\nu}^{k_{\nu-2}} [1] \\ \left\{ \begin{array}{l} w_1^{k_{\nu-1}} e^{\rho w_1 \pi} [1] \\ + a_{\nu-1} \rho^{\gamma_{\nu-1}} w_1^{k_{\nu-1}} [1] \end{array} \right\} & \left\{ \begin{array}{l} w_2^{k_{\nu-1}} e^{\rho w_2 \pi} [1] \\ + a_{\nu-1} \rho^{\gamma_{\nu-1}} w_2^{k_{\nu-1}} [1] \end{array} \right\} & \dots & \left\{ \begin{array}{l} w_{\nu}^{k_{\nu-1}} e^{\rho w_{\nu} \pi} [1] \\ + a_{\nu-1} \rho^{\gamma_{\nu-1}} w_{\nu}^{k_{\nu-1}} [1] \end{array} \right\} \\ \left\{ \begin{array}{l} w_1^{k_{\nu}} e^{\rho w_1 \pi} [1] \\ + a_{\nu} \rho^{\gamma_{\nu}} w_1^{k_{\nu}} [1] \end{array} \right\} & \left\{ \begin{array}{l} w_2^{k_{\nu}} e^{\rho w_2 \pi} [1] \\ + a_{\nu} \rho^{\gamma_{\nu}} w_2^{k_{\nu}} [1] \end{array} \right\} & \dots & \left\{ \begin{array}{l} w_{\nu}^{k_{\nu}} e^{\rho w_{\nu} \pi} [1] \\ + a_{\nu} \rho^{\gamma_{\nu}} w_{\nu}^{k_{\nu}} [1] \end{array} \right\} \end{vmatrix}.$$

³⁴ This does not correspond precisely to the determinant Δ_1 of the preceding section, as no exponential factor has been divided from the last rows.

Let ρ be restricted to the region T_0 . Suppose first, for simplicity, that the last two of the boundary conditions do not involve the point 0, so that $a_{\nu-1}$ and a_ν are zero. Then in each of the minors of the last two rows of the determinant, an exponential factor can be taken from each column, and the expansion according to these minors has the form

$$(40) \quad \sum \delta_{jj'} \zeta_{jj'} e^{\rho(w_j + w_{j'})\pi} [1],$$

where $\delta_{jj'}$ and $\zeta_{jj'}$ are determinants of $\nu - 2$ rows and 2 rows respectively, with powers of the quantities w_i as constituents. Of these determinants, δ_{12} , δ_{13} , ζ_{12} , and ζ_{13} are surely different from zero, having essentially the same form as δ_1 and δ_2 in the preceding section; it is important to observe that when the roots w_i are arranged in cyclic order, w_1 and w_2 are adjacent, and likewise w_1 and w_3 . It is immediately recognized that if ρ is in the remote part of T_0 , all the terms of (40) excepting possibly that with the subscripts 1, 3, are insignificant in comparison with the term that has the subscripts 1, 2, and may be merged with the latter term, by virtue of the latitude that is allowable in the interpretation of the bracket symbols. We may write

$$(41) \quad \Delta_1 = [\delta_{12}\zeta_{12}]e^{\rho(w_1 + w_2)\pi} + [\delta_{13}\zeta_{13}]e^{\rho(w_1 + w_3)\pi}.$$

This is on the assumption that $a_{\nu-1}$ and a_ν are zero. Among the terms which have to be added to those already taken into account, if this restriction is removed, the worst that can occur is $e^{\rho w_1\pi}$ multiplied by a power of ρ , with a further factor which remains finite. Such a term will be negligible in comparison with $e^{\rho(w_1 + w_2)\pi}$, provided that a power of ρ is negligible in comparison with $e^{\rho w_2\pi}$, which will be the case if the vertex w_2 of the polygon representing the roots remains at the right of the axis of imaginaries after the polygon has been rotated through an angle of π/ν , that is, if $2\pi/\nu < \frac{1}{2}\pi$, $\nu > 4$. As the condition last stated was imposed at the beginning of the section, the terms containing powers of ρ may be incorporated in the expression $[\delta_{12}\zeta_{12}]e^{\rho(w_1 + w_2)\pi}$, and the validity of (41) is general.

Suppose the bracket symbols in (41) replaced by their limiting values, and the resulting expression set equal to zero. Since

$$w_2 - w_3 = e^{\frac{\pi i}{\nu}} - e^{-\frac{3\pi i}{\nu}} = e^{-\frac{\pi i}{\nu}} \left(e^{\frac{2\pi i}{\nu}} - e^{-\frac{2\pi i}{\nu}} \right),$$

the equation so obtained, which obviously has only roots of the first order, is equivalent to the relation

$$(42) \quad 2\rho\pi i e^{-\frac{\pi i}{\nu}} \sin \frac{2\pi}{\nu} = \log \left(-\frac{\delta_{13}\zeta_{13}}{\delta_{12}\zeta_{12}} \right) + 2n\pi i,$$

where the first term on the right is a particular value of the logarithm, and n is any integer. As in the case previously treated, the logarithm is a pure imaginary. Consider, for example, the relation between the determinants δ_{12} and δ_{13} . The latter may be regarded as obtained from the former by replacing w_3 in the first column by w_2 . But it may also be regarded as obtained by replacing each of the roots from w_3 to w_ν by the root next to it cyclically in the negative direction (whereby w_3 goes into one of the roots with a subscript greater than 4, and w_4 goes into w_2) and then interchanging columns. This process amounts to a multiplication of each row of δ_{12} by a root of unity, and so to a multiplication of the whole determinant by a quantity of the same sort. Similar reasoning applies to ζ_{12} and ζ_{13} , and it follows that the ratio $\delta_{13}\zeta_{13}/\delta_{12}\zeta_{12}$ has its absolute value equal to unity, and the real part of its logarithm equal to zero. The successive values of ρ in (42) are not real, because of the factor $e^{-\pi i/\nu}$, but follow each other at equal intervals along a ray inclined to the axis of reals at an angle of π/ν . The correspondence between these values and the roots of the equation $\Delta_1 = 0$ is established as in the earlier cases.

The roots of $\Delta_1 = 0$ in the region $T_{2\nu-1}$ are similarly determined. There will be infinitely many of them, situated asymptotically along the ray $\arg \rho = -\pi/\nu$. But since the original problem involves ρ only in the ν th power, as has already been pointed out, the existence of roots ρ along this ray implies the existence of roots $\rho e^{2\pi i/\nu}$ which will be in the region T_0 , at least from a certain point on, and so will be among those already found. Thus the consideration of the region $T_{2\nu-1}$ adds only a finite number to the list of independent characteristic values. Hence:

The characteristic values of the system (15), (39), are simple roots of the determinant equation from a certain point on, and are expressible in the form ³⁵

$$\rho_n = \frac{\mu_0 + n + \epsilon_n}{\sin(2\pi/\nu)} e^{\frac{\pi i}{\nu}}, \quad n = g, g+1, g+2, \dots,$$

where μ_0 is real and independent of n , g is a suitable integer, positive, negative, or zero, and

$$\lim_{n=\infty} \epsilon_n = 0.$$

³⁵ It may be that there are infinitely many characteristic values just inside the lower border of $S_{2\nu-1}$, and then the formula will give instead of these the equivalent values in S_1 ; but the latter will serve as well and will be more con-

When n is large, a characteristic solution may be obtained by setting $\rho = \rho_n$ in the determinant Δ_1 , and replacing the t -th element of the last row by $y_t(x, \rho_n) = e^{\rho_n w_t x}$ [1], $t = 1, 2, \dots, \nu$; and this is except for a constant factor the only characteristic solution³⁶ for $\rho = \rho_n$. The numbers $\rho_n w_1$ are situated along the positive axis of reals, the numbers $\rho_n w_2$ and $\rho_n w_3$ along the rays making angles of $2\pi/\nu$ with the positive axis of reals, and so on. Hence, when the determinant is expanded according to the minors of its last two rows, two terms are predominant over the others when n is large and x is not equal to 0 or π , namely the terms involving³⁷ $e^{\rho_n(w_1\pi + w_2x)}$ and $e^{\rho_n(w_1\pi + w_3x)}$. The coefficients of these expressions are quantities which have already been seen not to vanish for large values of ρ . The limits approached by these coefficients are equal in absolute value, and if the whole determinant is divided by either square root of their product, they will be replaced by limits which are conjugate imaginaries. Dividing out also the factor $e^{\rho_n w_1 \pi}$, we may write

$$(43) \quad u_n(x) = [d_1] e^{\rho_n w_2 x} + [d_2] e^{\rho_n w_3 x},$$

where d_1 and d_2 are conjugate imaginary quantities different from zero,³⁸ and the limits indicated by the bracket symbols are approached uniformly in x for $x' \leq x \leq x''$, if $0 < x' < x'' < \pi$.

This expression for $u_n(x)$ is sufficient to suggest the proof of the first theorem, that a series of the form $\sum a_n u_n(x)$ which converges uniformly throughout an interval $0 \leq x < x_0$ must represent a function continuous there with its derivatives of all orders. In calculating the order of magnitude of the terms of the derived series it will be necessary to go behind the relation (43) to the complete determinant expression

venient. If the problem is real at the start, all the characteristic numbers from a certain point on will be actually on the ray $\arg \rho = \pi/\nu$, and not merely approach it asymptotically. For if ρ is a characteristic number, the conjugate of ρ is a characteristic number, and $e^{2\pi i/\nu}$ times the conjugate of ρ is a characteristic number, and unless $\arg \rho = \pi/\nu$ the number last obtained is distinct from ρ , and there are two characteristic values where, if ρ is large, there ought to be only one. It may be remarked that in the plane of the variable $\lambda = \rho^\nu$, the characteristic values in this case are real from a certain point on, and become *negatively* infinite.

³⁶ Cf. footnote in connection with the corresponding passage in the preceding section. It is readily shown that the solution indicated can not vanish identically.

³⁷ When $x = 0$ a number of terms are of the same order of magnitude, and when $x = \pi$ the terms involving $e^{\rho_n(w_1x + w_2\pi)}$ and $e^{\rho_n(w_1x + w_3\pi)}$ become comparable with those indicated above.

³⁸ They are not the same as the numbers d_1 and d_2 of the preceding section.

from which it was obtained; but the argument is so similar to the corresponding ones already given that its details may be omitted.

For the proof that the development of a particular analytic function is divergent, the form of the adjoint system is to be taken into account. The adjoint differential equation is of course the same as before. The boundary conditions may be discussed by an appropriate modification of the method used in the case of $\nu-1$ specialized conditions. It is found that $\nu-2$ of the adjoint conditions contain the point π only, while the remaining two not only may but surely do involve the point 0, in such a way that the sets of terms relating to this point in the two conditions are linearly independent.³⁹ The conditions may be simplified and arranged in a manner corresponding to our previous practice. The characteristic functions may be written in the form

$$(44) \quad v_n(x) = [d'_1] e^{\rho_n w_2(\pi - x)} + [d'_2] e^{\rho_n w_3(\pi - x)},$$

where the numbers ρ_n are those already determined, d'_1 and d'_2 are conjugate imaginary quantities different from zero, and the approach of the bracket functions to their limits is uniform throughout any closed interval interior to $(0, \pi)$. This expression, however, is not sufficient for the present purpose; for in determining the order of magnitude of the coefficients in the series it is precisely at the point $x = 0$, where the formula breaks down, that the value of $v_n(x)$ is needed. At this point, terms involving $e^{\rho_n(w_2\pi - w_1x)}$ and $e^{\rho_n(w_3\pi - w_1x)}$ must be explicitly taken into account.

The characteristic function (44) was obtained from the determinant corresponding to ⁴⁰ (33) by dividing out a factor $e^{\rho_n w_1 \pi}$, independent of x , and a constant factor depending on neither ρ nor x . By dividing out the same exponential factor, but a different constant factor, a characteristic function is obtained which may be represented adequately, in a sense presently to be explained, by the expression

$$\bar{v}_n(x) = \delta'_2 \begin{vmatrix} e^{-\rho_n w_1 x} & e^{\rho_n w_2(\pi - x)} \\ w_1 k'_2 & w_2 k'_2 e^{\rho_n w_2 \pi} \end{vmatrix} - \delta'_3 \begin{vmatrix} e^{-\rho_n w_1 x} & e^{\rho_n w_3(\pi - x)} \\ w_1 k'_2 & w_3 k'_2 e^{\rho_n w_3 \pi} \end{vmatrix},$$

³⁹ The latter part of this statement may be proved either by making use of the fact that the matrix corresponding to that which was called β must be non-singular, or by observing that if the statement is assumed to be untrue a direct determination of the characteristic values for the adjoint system leads to results inconsistent with those found for the given system.

⁴⁰ This determinant in the present case has elements of the form $w_i k'_2 e^{\rho_n w_i \pi} [1] + \beta'_2 \rho_n \gamma'_2 w_i k'_2 [1]$ in its second row, where k'_2 , γ'_2 , and κ'_2 are integers, of which γ'_2 may be negative, and β'_2 is a constant; cf. the determinant from which (43) was obtained.

consisting of the principal terms in the expansion of the modified determinant according to the minors of its first two rows; here δ'_2 and δ'_3 are quantities having the same absolute value, different from zero, and the constant factor just referred to is assumed to have been determined so that they are conjugate imaginaries, and k'_2 is the order of the highest derivative whose value at the point 0 enters into the second of the adjoint boundary conditions. Let us change our earlier notation to the extent of representing the new characteristic function, instead of (44), by $v_n(x)$. The claim that this function is sufficiently well represented by the function $\bar{v}_n(x)$ is to be understood as follows: Let $f(x)$ stand for the function (14), the integer q being left arbitrary for the present. If the bracket symbols are given the more specific interpretation explained in the latter part of the preceding section, and the various terms are integrated by parts, it is found that

$$(45) \quad \left| \int_0^\pi f(x) v_n(x) dx - \int_0^\pi f(x) \bar{v}_n(x) dx \right| < \frac{c}{n^{q+3}} e^{n\pi \cot(2\pi/\nu)},$$

where c is independent of n , like the quantities to be denoted by the same letter with a subscript in the remainder of this section. The value of $\int f(x) \bar{v}_n(x) dx$ differs by less than $c_1 e^{n\pi \cot(2\pi/\nu)} / n^{q+3}$ from

$$(46) \quad \frac{\delta'_2}{\rho_n^{q+2}} e^{\rho_n w_2 \pi} \begin{vmatrix} w_1^{-q-2} & w_2^{-q-2} \\ w_1^{k'_2} & w_2^{k'_2} \end{vmatrix} - \frac{\delta'_3}{\rho_n^{q+2}} e^{\rho_n w_3 \pi} \begin{vmatrix} w_1^{-q-2} & w_3^{-q-2} \\ w_1^{k'_2} & w_3^{k'_2} \end{vmatrix},$$

in comparison with which expression the difference just mentioned and the quantity (45) are insignificant, *unless the four terms obtained by expanding the two determinants nearly or quite destroy each other*. It is to be shown that this mutual destruction will not take place, at least if q is suitably chosen.

If the first determinant is divided by $w_2^{-q-2+k'_2}$, and the second by $w_1^{-q-2+k'_2}$, the remaining factors will be equal in the two cases, except as to sign, since w_1/w_2 is the same as w_3/w_1 , each being equal to $e^{-2\pi i/\nu}$. The common factor $e^{2(q+2)\pi i/\nu} - e^{-2k'_2\pi i/\nu}$ will vanish only if $-q-2$ is congruent to $k'_2 \pmod{\nu}$, that is, for only one of any ν consecutive values of q . On the other hand, the expression

$$(47) \quad \frac{1}{\rho_n^{q+2}} \left(\delta'_2 w_2^{-q-2+k'_2} e^{\rho_n w_2 \pi} + \delta'_3 w_1^{-q-2+k'_2} e^{\rho_n w_3 \pi} \right),$$

which forms the remaining factor of (46), can be treated as the expression in braces in (37) was treated; in any ν consecutive values of q there can not be more than one for which its absolute value fails to

exceed a quantity $c_2 e^{n\pi \cot(2\pi/\nu)} / n^{q+2}$. As $\nu \geq 5$, by hypothesis, there will be at least three of any ν successive values of q for which

$$\left| \int_0^\pi f(x) v_n(x) dx \right| > \frac{c_3}{n^{q+2}} e^{n\pi \cot(2\pi/\nu)}.$$

It is readily seen that

$$\left| \int_0^\pi u_n(x) v_n(x) dx \right| < c_4 e^{n\pi \cot(2\pi/\nu)},$$

and the rest of the divergence proof follows closely the lines of those already indicated.

It suffices now to give a brief description of the principal features of the problem which is presented if μ of the given boundary conditions, instead of two, involve the point π , the number μ being any number less than half as large as ν . The subscripts of the roots w_i are to be assigned in such a way that if θ is between 0 and π/ν the real parts of the numbers $w_i e^{\theta i}$ succeed each other in decreasing order of algebraic magnitude. In the expansion of the determinant from which the characteristic values are found, the largest exponentials will be common to the principal terms, and so will not have an influence on the vanishing of the function; the occurrence of roots will be rendered possible by the balancing of $e^{\rho w_\mu \pi}$ against $e^{\rho w_{\mu+1} \pi}$, when μ is odd, and by the balancing of $e^{\rho w_\mu \pi}$ against $e^{\rho w_{\mu+1} \pi}$ on the one hand and of $e^{\rho w_{\mu-1} \pi}$ against $e^{\rho w_{\mu+1} \pi}$ on the other, when μ is even. The characteristic values of sufficiently large index will be roots of the first order, and will occur at approximately equal intervals; for odd values of μ they will be distributed asymptotically along the axis of reals, and for even values of μ , along the ray $\arg \rho = \pi/\nu$. They will be of amplitude 0 or π/ν exactly, from a certain point on, if the problem is real at the start. The characteristic functions may be represented by the expression

$$(48) \quad u_n(x) = [d_1] e^{\rho_n w_\mu x} + [d_2] e^{\rho_n w_{\mu+1} x},$$

where d_1 and d_2 are conjugate imaginary quantities different from zero, and the asymptotic representation holds uniformly throughout any closed interval interior to $(0, \pi)$. It will be observed that the numbers $\rho_n w_\mu$ and $\rho_n w_{\mu+1}$ are situated asymptotically along rays making angles of $\mu\pi/\nu$ with the positive axis of reals, whether μ is even or odd. By using a somewhat more explicit form for the characteristic functions, it may be shown that if a series of constant multiples of them con-

verges uniformly, its sum must necessarily have continuous derivatives of all orders.

Of the adjoint boundary conditions, $\nu - \mu$ depend on the point π only, while the remaining μ contain linearly independent combinations of the values of v and its derivatives at the point 0. The characteristic functions of the adjoint system have an expression

$$v_n(x) = [d'_1] e^{\rho_n w_\mu(\pi-x)} + [d'_2] e^{\rho_n w_{\mu+1}(\pi-x)},$$

the interpretation of which is similar to that of (48). Finally, the general coefficient in the formal development of the function (14) may be treated as in the previous cases, and it may be shown that for infinitely many positive integral values ⁴¹ of q the series diverges at every interior point of the interval $(0, \pi)$.

It has been pointed out that the numbers $\rho_n w_\mu$ and $\rho_n w_{\mu+1}$, which figure in the formula (48), are situated along rays which become less and less inclined to the axis of pure imaginaries as μ increases. If an analogous formula were to hold in the case that ν is even and $\mu = \frac{1}{2}\nu$, which has been excluded by our hypotheses, the corresponding numbers would be distributed along the axis of imaginaries itself, and the formula would resemble those which Birkhoff finds in the case of regular boundary conditions.⁴² As a matter of fact, the boundary conditions do become regular when $\mu = \frac{1}{2}\nu$, if the order of the derivatives at the point 0 involved in the last μ conditions is not too high. This is precisely what happens in the case of Liouville's conditions, with $\mu = 1$, when ν is equal to 2.

⁴¹ In the formula corresponding to (46), there will be a determinant containing $w_1, \dots, w_{\mu-1}, w_\mu$, and one containing $w_1, \dots, w_{\mu-1}, w_{\mu+1}$. When μ is odd, a common factor is disclosed by dividing the first determinant by a power of w_1 , and the second by the like power of w_2 ; when μ is even, the second determinant is to be divided by a power of w_1 , and the first by this power of w_2 . Among ν successive values of q , there will be just $\mu - 1$ values for which the determinants vanish. Not more than one other value is excluded by the requirement as to the order of magnitude of the quantity corresponding to (47), and there remain at least $\nu - \mu$ of every ν consecutive values of q , which are available for the purposes of the demonstration.

⁴² In the paper II, p. 389. The analogy is not perfect; the determination of the characteristic values becomes less simple in the regular case.